Quantum Mechanics-Extra questions

Phase velocity of de-Broglie waves exceeds velocity of light

- According to de-Broglie, a moving particle of mass m and velocity v, is associated with a single wave. The phase velocity is given by

$$v_p = \lambda\nu$$

$$E = h\nu \text{ or } \quad \nu = \frac{E}{h}$$

$$\lambda = \frac{h}{mv}$$

$$v_p = \lambda\nu = \frac{h}{mv} \cdot \frac{E}{h}$$

$$\mathbf{v_p = \frac{E}{mv}}$$

- From Einstein's equation: $E=mc^2$

$$v_p = \frac{mc^2}{mv} = \frac{c^2}{v}$$

By theory of relativity no particle can move with speed of light. Hence, $c > v$

It clearly indicates that phase velocity v_p of the associated wave is greater than c, velocity of light. It indicates that the associated wave with the particle moves faster than the particle itself. The particle will be left far behind. The phase velocity is a purely mathematical concept and represents the rate at which a given phase of a monochromatic wave moves forward.

Relation between v_g and v_p for non-relativistic free particle

- The phase velocity of the wave associated with non-relativistic free particle

$$v_p = \vartheta\lambda \ldots\ldots (1)$$

According to de-Broglie hypothesis

$$\lambda = \frac{h}{mv_g} \ldots\ldots\ldots (2)$$

Energy of free particle $E = \frac{1}{2}mv_g^2$

$$\vartheta = \frac{E}{h} = \frac{\frac{1}{2}mv_g^2}{h}$$

From eqn.(1) and (2)

$$v_p = \frac{\frac{1}{2}mv_g^2}{h} \cdot \frac{h}{mv_g}$$

$$\mathbf{v_p = \frac{v_g}{2}}$$

Group velocity is the particle velocity

(1) For non-relativistic particle

$$E=\frac{1}{2}mv^2=\frac{(mv)^2}{2m}=\frac{(p)^2}{2m}$$

Hence, $\frac{dE}{dp}=v$ ....(1)

$E=\hbar w$ ....(2)

From de-Broglie concept, $p=\frac{h}{\lambda}$

$p=\hbar k$ ....(3)

From (2) and (3)

$dE=\hbar\, dw$

$dp=\hbar\, dk$

Hence, $\frac{dE}{dp}=\frac{\hbar dw}{\hbar dk}=\frac{dw}{dk}=v_g$; group velocity.....(4)

From (1) and (4)

$$\mathbf{v_g=v}$$

(2) For a relativistic particle

Energy and momentum of the particle are connected as

$E^2 = p^2 c^2 + m_0^2 c^2$

$$\frac{dE}{dp} = \frac{pc^2}{E}$$

Relativistically, $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$ and momentum $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\frac{dE}{dp} = \frac{pc^2}{E} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \cdot c^2 \Big/ \left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) = v$$

$$\frac{dE}{dp} = v$$

As already proved:

$$dE = \hbar\, dw$$

$$dp = \hbar\, dk$$

Hence, $\frac{dE}{dp} = \frac{\hbar dw}{\hbar dk} = \frac{dw}{dk} = v_g$; group velocity

Thus, $v = v_g$ for relativistic particle