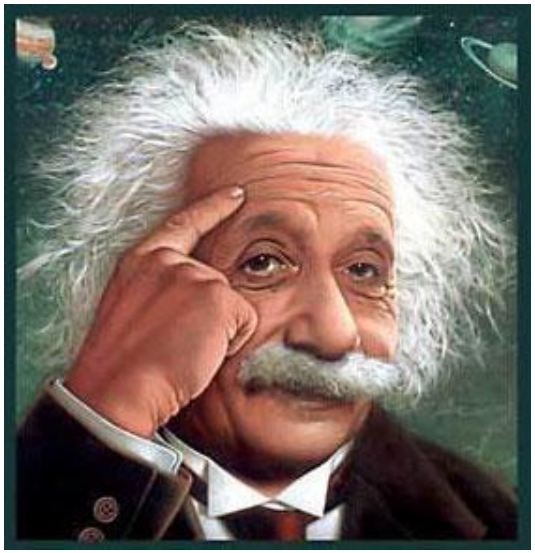
UNIT-IV

Special Theory of Relativity

Albert Einstein



In 1905, Albert Einstein developed his Special Theory of Relativity

- **The Special Theory of relativity:**

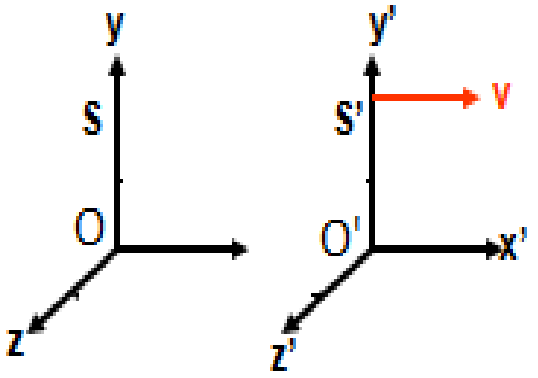
  It deals only with the description of events by the observers in inertial reference frames

- **The General Theory of Relativity:**

  It concerns itself with all reference frames, including non-inertial frames.

Some important definitions

- **Event:** A sudden occurrence of a happening in space at a given time
- **Observer:** Anything (person or instrument) which measures the event.
- **Frame of reference:** A system of co-ordinates fixed to measure the event.



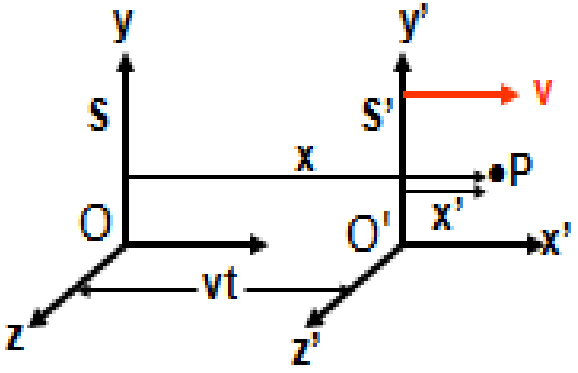
- **Inertial frame of reference:** A frame of reference in which Newton's law of motion are applicable. Every frame of reference either at rest or in uniform motion are inertial

- **Non-Inertial frame of reference:** Newton's law of motion are no longer valid in such frames.

Earth rotates around the sun and about its on axis with an angular acceleration very-2 small so for practical purpose, it can be assumed as inertial frame of reference

Galilean Transformations

According to the principle of Galilean relativity, the laws of mechanics must be the same in all inertial frames of reference.



Let v = velocity with which S' moving w.r.t S along x – axis

After time t, distance moved by S' = vt

Galilean Transformations

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Inverse Galilean Transformations

$$x = x' + vt$$
$$y = y'$$
$$z = z'$$
$$t = t'$$

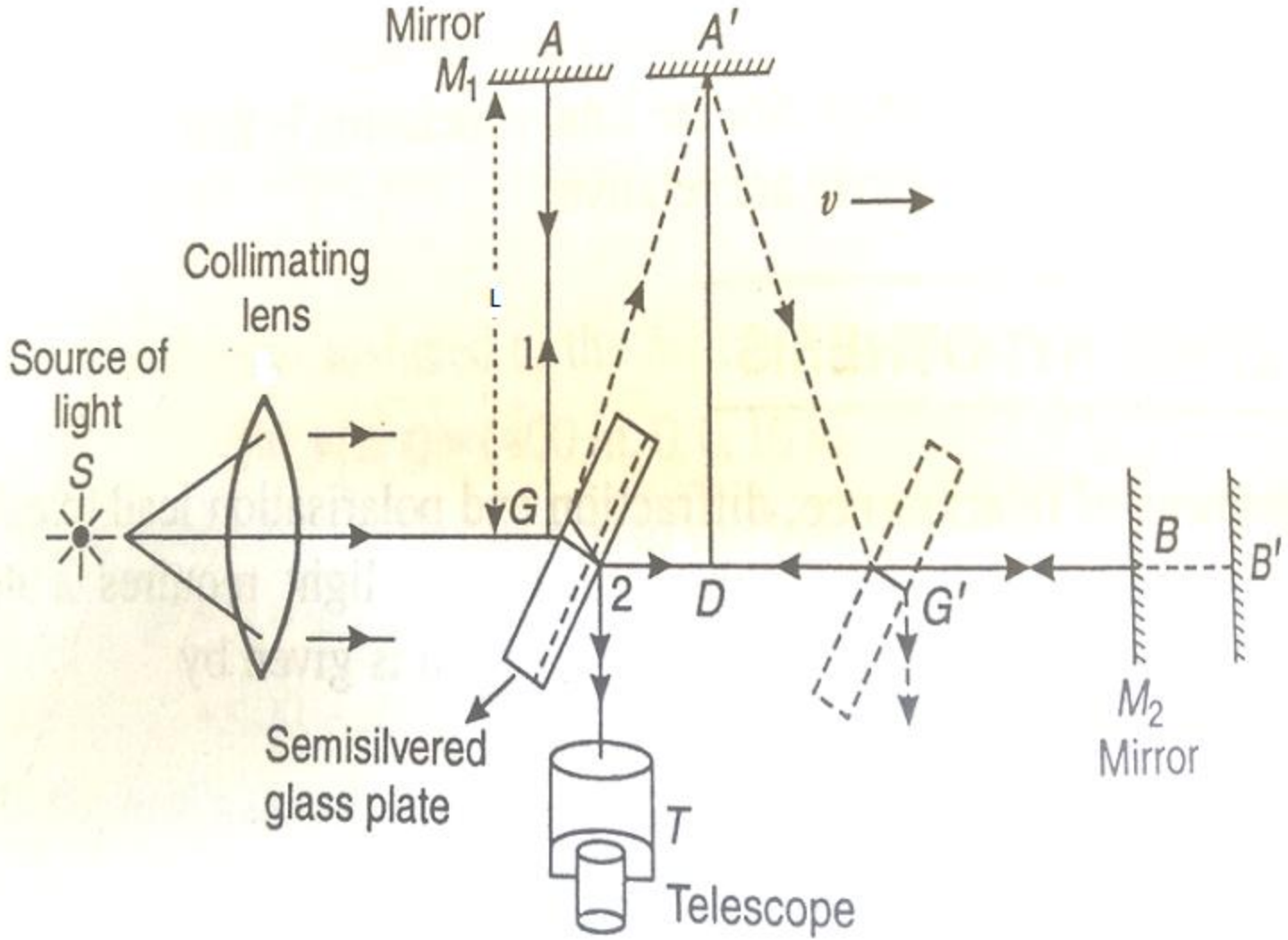
Michelson-Morley Experiment

This experiment was performed by Michelson and Morley to prove the existence of ‘ether’. The experiment is based upon interference of light waves.

Michelson Interferometer was used for this experiment. One arm is parallel to the motion of the Earth other arm is in perpendicular direction.

Light from source S is split by semi-silvered glass plate and travels to mirrors in mutually perpendicular directions

After reflection the beams recombine slightly out of phase due to the “ether wind” as viewed by telescope T.


Mirror
M_1
A
A′
v
Collimating
lens
Source of
light
S
L
1
G
2
D
G′
B
B′
M_2
Mirror
Semisilvered
glass plate
T
Telescope

The time taken by reflected ray of light $t_1 = 2L/\sqrt{(c^2-v^2)}$

The time taken by transmitted ray of light $t_2 = 2Lc/(c^2-v^2)$

The time difference between two rays reaching at observer is given by

$$\Delta t = Lv^2/c^3$$

And the path difference is given by $\Delta x = c\,\Delta t = Lv^2/c^2$

If the whole system is rotated through 90° then maximum path difference = $2\,Lv^2/c^2$

The condition of bright fringes is given by

$\Delta x = 2Lv^2/c^2 = n\lambda$ $\qquad\qquad$ $n = (2Lv^2/c^2)/\lambda$

In Michelson Morley experiment

L=1m, v = 3×10^4 m/s and wavelength of light = 5.5×10^{-7} m

The fringe shift (n)=0.4

Experimentally no fringe shift was observed in the experiment even when the experiment was performed during day/night, during different seasons and at different places. The experiment concluded that there is no relative motion between earth and ether

Interpretation of negative results

- Michelson's view:

  The ether is dragged along the earth due which the relative motion may not be detected.

- Fitzgerald and Lorentz View:

  Bodies contract along the direction of motion by a factor $\sqrt{(1-v^2/c^2)}$

- Einstein's View:

  Velocity of light in space is universal constant, so can not be taken relative.

Einstein Special theory of relativity

After the failure of Michelson Morley Experiment Einstein proposed two postulates of his special theory of relativity:

1. All the laws of physics are same in all inertial frames of reference either at rest or in uniform motion

2. The speed of light in the free space is same for all observer regardless of their state of motion

Lorentz transformation Equations

- Based upon theory of relativity, Lorentz derived transformation equations given by

Lorentz Transformation

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

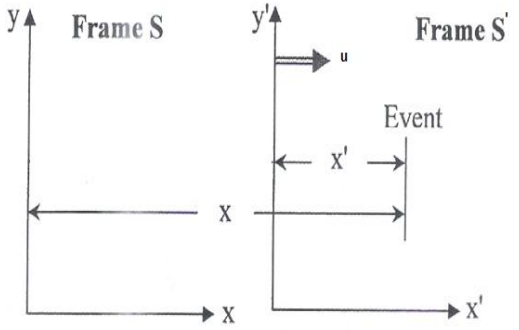
Lorentz Inverse Transformation

$$x = \frac{x' + ut'}{\sqrt{1 - u^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + ux'/c^2}{\sqrt{1 - u^2/c^2}}$$



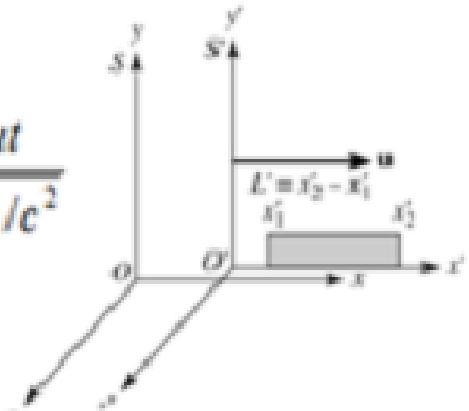
Length Contraction (Lorentz-Fitzgerald contraction)

Let us consider two frames of reference S and S' such that S' is moving with a uniform velocity u w.r.t. frame S along X-axis. Let consider any object (AB) observed by observer from the two frames S & S'.

Measurement of positions at endpoints at same time in frame S' and in frame S, gives the length $L' = x_2' - x_1'$ using

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}$$

$$x_2' - x_1' = \frac{(x_2 - x_1) - u(t_2 - t_1)}{\sqrt{1 - u^2/c^2}}$$



To make the Observer in reference frame S to be able to measure the length of the object in S', he should mark the two ends of the object at same instant of time i.e. $t_2 = t_1$

$$L' = \frac{L}{\sqrt{1 - u^2/c^2}}$$

$$L_{moving} = L_{rest}\sqrt{1 - u^2/c^2} = L_{rest}/\gamma$$

where, $\gamma = 1/\sqrt{(1 - u^2/c^2)}$

Example:

Calculate the apparent length of a metre rod, when it is carried in a rocket at a speed of 2.8×10^8 ms^{-1}. Take $c = 3 \times 10^8$ ms^{-1}.

Solution. Here $l_0 = 1$ m, $v = 2.8 \times 10^8$ ms^{-1} and $c = 3 \times 10^8$ ms^{-1}

Let l be the apparent length.

We know that,

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$= 1 \sqrt{1 - \left(\frac{2.8 \times 10^8}{3 \times 10^8}\right)^2}$$

$$= \sqrt{0.128} = 0.357 \text{ m}$$

$\therefore$ Apparent length is 0.357 m.

Time Dilation (Relativity of time)

Let t_1 and t_2 be the instant of time when first and second events (say successive ticking of clock or two consecutive heart beats) are recorded by a clock at position x in frame S which is at rest.

Similarly, let t_1' and t_2' be the instants recorded by a clock in frame S' moving with relative velocity v, for the same two successive events taking place at position x. Then,

$$t_1' = \frac{t_1 - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t_2' = \frac{t_2 - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \quad t_2' - t_1' = \left(\frac{t_2 - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) - \left(\frac{t_1 - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

or
$$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

If $t_2 - t_1 = T_0$ and $t_2' - t_1' = T$ are time elapsed between two given events measured by clock in frame S and S′ respectively, then we get

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

As $v < c$, therefore $T > T_0$. Hence the observer in frame S′ measures longer time interval between two events as compared to the time interval measured by same clock between the same two events in frame S which is at rest *i.e.* a clock in frame S appears to go slow to an observer in S′ which is in motion relative to S. This is called **time dilation.**

Find the speed at which a clock should move relativ to an observer so that it appears to lose 1 minute per day.

Solution. Here time measured by the clock T = 24 × 60 min

Time measured by the observer when the clock is in motion

$$T_0 = (24 \times 60 - 1) \text{ minute}$$

Let v be velocity of clock, then $T = \dfrac{T_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

or $$1 - \frac{v^2}{c^2} = \left(\frac{T_0}{T}\right)^2$$

$$1 - \frac{v^2}{c^2} = \left[\frac{24 \times 60 - 1}{24 \times 60}\right]^2 = \left(\frac{1439}{1440}\right)^2 = (0.9993)^2$$

$$\frac{v^2}{c^2} = 1 - (0.9993)^2 = 1 - 0.9986 = 0.0014$$

$$\frac{v}{c} = 0.0374$$

$$v = 0.0374c$$

$$v = 0.0374 \times 3 \times 10^8 = \mathbf{0.1122 \times 10^8\ ms^{-1}}.$$

Lorentz velocity transformations (Addition of Velocities)

Consider two frames S and S' such that S' is moving with velocity v relative to S along x-axis. Suppose a particle moves with velocity components u_x , u_y, u_z relative to S and we have to find velocity components u'_x , u'_y, u'_z relative to S'.

According to Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad \ldots(1)$$

and

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}} \qquad \ldots.(2)$$

Differentiate equations (1) and (2) we get

$$dx' = \frac{dx - v\,dt}{\sqrt{1 - v^2/c^2}} \qquad \ldots.(3)$$

and

$$dt' = \frac{dt - \frac{v\,dx}{c^2}}{\sqrt{1 - v^2/c^2}} \qquad \ldots.(4)$$

As $$u'_x = \frac{dx'}{dt'}$$

$$\therefore \quad u'_x = \frac{dx - \upsilon dt}{dt - \frac{\upsilon dx}{c^2}} = \frac{\frac{dx}{dt} - \upsilon}{1 - \frac{\upsilon}{c^2}\frac{dx}{dt}}$$

or $$u'_x = \frac{u_x - \upsilon}{1 - \frac{u_x \upsilon}{c^2}}$$

Similarly we can find $u'_{y,}$ and u'_z

Special case:

If particle moving in frame S' is photon, then we have

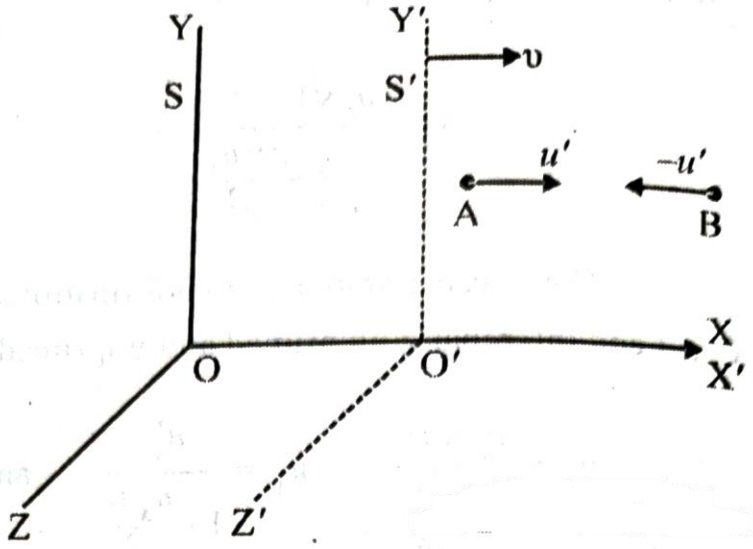
$$u'_x = c$$

From $$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

we have $$u_x = \frac{c + v}{1 + \frac{cv}{c^2}} = c = u'_x$$

Variation of mass with velocity (Relativity of mass)

Consider two frames S and S', where S' is moving with velocity v relative to S along X-axis, as shown in figure. Consider two identical bodies A and B are approaching each other with velocity u' in frame S'. After collision, the two bodies coalesced into one body and comes to rest in frame S'.



Let us analyse this collision w.r.t. stationary frame S. Suppose u_1, u_2 be the velocities of the bodies A and B relative to S, then using law of addition of velocities, we have

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}} \qquad \text{....(1)}$$

and $$u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}} \qquad \text{....(2)}$$

The coalesced body of mass $(m_1 + m_2)$ is at rest in frame S', but have velocity v relative to S.

According to principle of conservation of momentum

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\,\upsilon$$

or
$$m_1\left(\frac{u' + \upsilon}{1 + \frac{u'\upsilon}{c^2}}\right) + m_2\left(\frac{-u' + \upsilon}{1 - \frac{u'\upsilon}{c^2}}\right) = (m_1 + m_2)\,\upsilon$$

or
$$m_1\left(\frac{u' + \upsilon}{1 + \frac{u'\upsilon}{c^2}} - \upsilon\right) = m_2\left(\upsilon - \frac{-u' + \upsilon}{1 - \frac{u'\upsilon}{c^2}}\right)$$

or
$$m_1\left(\frac{u' + \upsilon - \upsilon - \frac{u'\upsilon^2}{c^2}}{1 + \frac{u'\upsilon}{c^2}}\right) = m_2\left(\frac{\upsilon - \frac{u'\upsilon^2}{c^2} + u' - \upsilon}{1 - \frac{u'\upsilon}{c^2}}\right)$$

or
$$\frac{m_1}{m_2} = \frac{1 + \frac{u'\upsilon}{c^2}}{1 - \frac{u'\upsilon}{c^2}} \qquad \text{....(3)}$$

Now from equation (1), we have

$$1-\frac{u_1^2}{c^2}=1-\frac{1}{c^2}\left(\frac{u'+v}{1+\frac{u'v}{c^2}}\right)^2=\frac{\left(1+\frac{u'v}{c^2}\right)^2-\frac{1}{c^2}(u'+v)^2}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$=\frac{1+\frac{u'^2v^2}{c^4}+\frac{2u'v}{c^2}-\frac{u'^2}{c^2}-\frac{v^2}{c^2}-\frac{2u'v}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$=\frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u'^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2}=\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2} \qquad \text{....(4)}$$

Similarly, we have

$$1-\frac{u_2^2}{c^2}=\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u'v}{c^2}\right)^2} \qquad \text{....(5)}$$

Dividing equations (5) by (4), we get

$$\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}=\frac{\left(1+\frac{u'v}{c^2}\right)^2}{\left(1-\frac{u'v}{c^2}\right)^2}$$

or

$$\frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}=\frac{\left(1+\frac{u'v}{c^2}\right)}{\left(1-\frac{u'v}{c^2}\right)} \qquad \text{....(6)}$$

(3) and (6), we get

$$\frac{m_1}{m_2} = \frac{\sqrt{1-u_2^2/c^2}}{\sqrt{1-u_1^2/c^2}} \qquad \ldots(7)$$

If body B is at rest before collision w.r.t. frame S, then $u_2 = 0$ and $m_2 = m_0$ i.e. rest mass of body.

Therefore, from equation (7), we get

$$\frac{m_1}{m_0} = \frac{1}{\sqrt{1-u_1^2/c^2}} \qquad \text{or} \qquad m_1 = \frac{m_0}{\sqrt{1-u_1^2/c^2}}$$

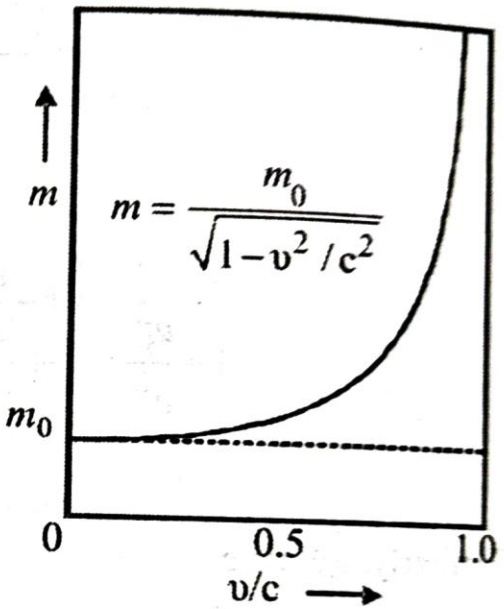
As two bodies are identical, so rest mass of another body (i.e. body A) is also m_0. Therefore, above equation may be applied to a single body of rest mass m_0 and effective mass m moving with velocity υ.

$$\therefore \qquad m = \frac{m_0}{\sqrt{1-\upsilon^2/c^2}}$$

Thus according to relativistic mechanics, mass m of a body is not constant but varies with velocity.

The variation of mass with velocity is as shown in figure.

If velocity is low i.e. $\upsilon << c$, then υ/c is small and υ^2/c^2 can be neglected as compared to 1 and hence $m = m_0$ i.e. mass does not change.



Example:

Calculate the speed of an electron at which it should move so that its relativistic mass becomes $\sqrt{3}$ times its rest mass.

Solution. Mass of a body moving with velocity v is

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}, \text{ where } m_0 \text{ is the rest mass}$$

Here, $m = \sqrt{3}\, m_0$

$$\therefore \quad \sqrt{3}\, m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \quad \sqrt{3}\sqrt{1-\frac{v^2}{c^2}} = 1 \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\sqrt{3}} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{3}$$

$$\Rightarrow \quad 1-\frac{1}{3} = \frac{v^2}{c^2} \Rightarrow \frac{2}{3} = \frac{v^2}{c^2}$$

$$\Rightarrow \quad v^2 = \frac{2}{3} \times c^2 = \frac{2}{3} \times 9 \times 10^{16} = 6 \times 10^{16}$$

$$\therefore \quad v = 2.45 \times 10^8 \text{ ms}^{-1}.$$

MASS-ENERGY EQUIVALENCE RELATION

Consider the example of fire produced from wood burning, the mass of ash is less than the mass of wood.

What are the source of light and heat energy in combustion process?

What are the source of nuclear energy?

Answer is given in Einstein theory of relativity

According to mass-energy equivalence mass is converted into energy and vice versa according to relation

$$E = mc^2$$

EINSTEIN MASS ENERGY $E = mc^2$

Consider a particle of mass m being acted upon by a force F such that it gains a velocity v. The increase in energy of the particle is defined in term of work done on the particle. If F is the applied force which displaces the particle through a small distance ds, then the work done is given by

$$dw = F \cdot ds \tag{1}$$

From Newton Second Law

$$F = \frac{dP}{dt} = \frac{d}{dt}(mv) \tag{2}$$

Since mass varies with velocity. It cannot be considered as a constant. Thus we obtain

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} \tag{3}$$

Substituting the value of F from Eq. (3) to Eq. (1)

$$dw = \left(m\frac{dv}{dt} + v\frac{dm}{dt}\right) \cdot ds$$

dw = Increase in K.E. = dk

$$\begin{aligned} \text{K.E.} &= m\frac{dv}{dt} \cdot ds + v\frac{dm}{dt} \cdot ds \\ &= m\frac{ds}{dt} \cdot dv + v\frac{ds}{dt} \cdot dm \\ &= mv \cdot dv + v \cdot v \cdot dm \\ &= mv\,dv + v^2 dm \end{aligned} \tag{4}$$

From special theory of relativity, we know that

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

where m_0 is rest mass of body while m is its mass during its motion.

$$\therefore \quad m^2 = \frac{m_0^{\,2}}{1-\frac{v^2}{c^2}}$$

or
$$m^2 = \frac{m_0^{\,2}\,c^2}{c^2 - v^2}$$

or
$$m^2c^2 - m^2v^2 = m_0^{\,2}\,c^2$$

Differentiating the above relation on both sides, we get

$$2\,m\,dm.c^2 - 2mdmv^2 - 2vdv.m^2 = 0 \qquad [\because m_0 \text{ and } c \text{ are constants}$$

or
$$dm\,c^2 = mv\,dv + v^2\,dm \qquad (5)$$

Comparing eqn. (4) and (5), we get

$$dE_k = dm\,c^2$$

Integrating both sides, we have

$$c^2 \int_{m_0}^{m} dm = \int dE_k$$

[$\because$ *Mass of body varies between* m_0 *and* m]

$$c^2 (m - m_0) = E_k$$

$$E_k = c^2 \times \Delta m$$

where $\Delta m = (m - m_0)$, is increase in mass which is converted into energy

$$E_k = \Delta m.c^2$$

As m_0 is rest mass of body therefore m_0c^2 is taken as the rest mass energy of the body which is the internal energy of the body.

$\therefore$ Total energy of body = K.E. of body + rest mass energy

$$E = E_k + m_0c^2$$

$$E = (m - m_0)\, c^2 + m_0c^2$$

$$E = mc^2$$

This relation is known as Einstein's mass-energy relationship.

Example:

Calculate the heat energy in calories when 10 gm of a substance is fully converted into energy.

Solution. Here, $m_0 = 10 \text{ gm} = 10^{-2} \text{ kg}$

Energy released on conversion of mass $E = m_0 c^2$

or $$E = 10^{-2} \times (3 \times 10^8)^2 = 9 \times 10^{14} \text{ J}$$

$$\therefore \quad \text{Energy in calories} = \frac{9 \times 10^{14}}{4.2} = 2.143 \times 10^{14} \text{ calories.}$$

Energy-Momentum Relation

RELATION BETWEEN RELATIVISTIC ENERGY AND MOMENTUM

According to relativistic mechanics, total energy of a body is given as

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\upsilon^2/c^2}}$$

As momentum of body is given as $p = m\upsilon$, so $\upsilon = \frac{p}{m}$

Therefore,
$$E = \frac{m_0 c^2}{\sqrt{1-\frac{p^2}{m^2 c^2}}} = \frac{m_0 c^2}{\sqrt{1-\frac{p^2 c^2}{m^2 c^4}}}$$

$$E = \frac{m_0 c^2}{\sqrt{1-p^2 c^2/E^2}} \qquad \left[\because E = mc^2\right]$$

Squaring both sides of above equation

$$E^2 = \frac{m_0^2 c^4}{1 - p^2 c^2 / E^2}$$

$$E^2 \left(1 - p^2 c^2 / E^2\right) = m_0^2 c^4$$

$$E^2 - p^2 c^2 = m_0^2 c^4$$

$$E^2 = p^2 c^2 + m_0^2 c^4$$

or $$E = \sqrt{p^2 c^2 + m_0^2 c^4}$$

Momentum of photon: As rest mass of photon is zero i.e. $m_0 = 0$, therefore above equation reduces to

$E^2 = p^2 c^2 + 0$

$$E = pc$$

or $$p = \frac{E}{c}$$

Examples of mass energy equivalence

1. Pair production:

Pair of electron and positron are formed when gamma radiation of energy 1.02 Mev interact with nucleus.

2. Annihilation:

A positron and electron combined with each other and a photon of energy 1.02 MeV is produced.

3. Mass defect:

During a nuclear reaction the mass of initial reactants is more than that of product. The difference of mass is called mass defect. Which is converted into energy during nuclear reaction (fission, fusion etc.)